

SINKHORN-KNOPP THEOREM FOR POSITIVE MAPS

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ABSTRACT. A positive map $S : M_k \rightarrow M_m$ is called doubly stochastic if $S(\frac{Id}{\sqrt{k}}) = \frac{Id}{\sqrt{m}}$ and $S^*(\frac{Id}{\sqrt{m}}) = \frac{Id}{\sqrt{k}}$. Here, we show that, for a positive map $T : M_k \rightarrow M_m$, there are invertible matrices $X' \in M_k, Y' \in M_m$ such that $Y'T(X'XX'^*)Y'^*$ is doubly stochastic if and only if $T(Id)$ and $T^*(Id)$ are invertible matrices and there are orthogonal projections $V_i \in M_k, W_i \in M_m, 1 \leq i \leq s$, such that $\mathbb{C}^k = \bigoplus_{i=1}^s \Im(V_i), \mathbb{C}^m = \bigoplus_{i=1}^s \Im(W_i)$ and, for every i , $T(V_i M_k V_i) \subset W_i M_m W_i, \frac{\text{rank}(W_i)}{\text{rank}(V_i)} = \frac{m}{k}$ and $\text{rank}(X)\text{rank}(W_i) < \text{rank}(T(X))\text{rank}(V_i)$ for every positive semidefinite Hermitian matrix $X \in V_i M_k V_i$ with $0 < \text{rank}(X) < \text{rank}(V_i)$. In order to obtain this theorem, we generalize Sinkhorn and Knopp ideas of support and total support for positive maps and we adapt their iterative algorithm. This result provides a necessary and sufficient condition for the existence of the filter normal form, which is commonly used in Quantum Information Theory. We prove the existence of this normal form for states $A \in M_k \otimes M_m \simeq M_{km}$ such that $\dim(\ker(A)) < k - 1$, if $k = m$, and $\dim(\ker(A)) < \min\{k, m\}$, if $k \neq m$.

INTRODUCTION

The Sinkhorn-Knopp theorem says that there are positive diagonal matrices D_1, D_2 such that $D_1 M D_2$ is doubly stochastic if and only if the square matrix M with non-negative entries has total support. In [14], the authors provide an iterative algorithm in order to obtain the doubly stochastic matrix from the original matrix. The convergence of this algorithm was proved, whenever the original matrix has support.

There are generalizations of Sinkhorn-Knopp theorem [1, 3, 5, 6, 8, 10, 13, 15]. One of them is particularly important for Quantum Information Theory, it is the so-called filter normal form (see [6, section IV.D]). Let us identify the tensor product space $M_k \otimes M_m$ with M_{km} , via Kronecker product, where M_k denotes the set of complex matrices of order k . Let a linear map $T : M_k \rightarrow M_m$ be positive, if it sends positive semidefinite Hermitian matrices to positive semidefinite Hermitian matrices. A positive map $T : M_k \rightarrow M_m$ is called doubly stochastic if $T(\frac{Id}{\sqrt{k}}) = \frac{Id}{\sqrt{m}}$ and $T^*(\frac{Id}{\sqrt{m}}) = \frac{Id}{\sqrt{k}}$, where T^* is the adjoint of T with respect to the trace inner product. Let us say that a positive map $T : M_k \rightarrow M_m$ is equivalent to a doubly stochastic map, if there are invertible matrices $X' \in M_k, Y' \in M_m$ such that $Y'T(X'XX'^*)Y'^*$ is doubly stochastic. Let $\text{tr}(C)$ denote the trace of $C \in M_k$.

The filter normal form established in [15] (and later in [10]) says that for a positive definite Hermitian matrix $D \in M_k \otimes M_m \simeq M_{km}$, there are invertible matrices $X' \in M_k, Y' \in M_m$ such that $(X' \otimes Y')D(X' \otimes Y')^* = \sum_{i=1}^n C_i \otimes D_i, C_1 = \frac{Id}{\sqrt{k}}, D_1 = \frac{Id}{\sqrt{m}}$ and $\text{tr}(C_i C_j) = \text{tr}(D_i D_j) = 0$, for every $i \neq j$.

The exact conditions for an arbitrary positive semidefinite Hermitian matrix $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m \simeq M_{km}$ to be put in the filter normal form are unknown. However, we can show the following: A can be put in the filter normal form if and only if the positive map $G_A : M_k \rightarrow M_m$, defined by $G_A(X) = \sum_{i=1}^n B_i \text{tr}(A_i X)$, is equivalent to a doubly stochastic map (see theorems 3.1 and 2.12).

The problem of determining whether a positive map $T : M_k \rightarrow M_k$ (the case $k = m$) is equivalent to a doubly stochastic one has been proved to be equivalent to a fixed point problem in [7]. Some

sufficient conditions for the existence of this fixed point were obtained there. For example, if $T(X)$ is positive definite, whenever X is a non null positive semidefinite Hermitian matrix then T is equivalent to a doubly stochastic map (See also [5]). This result provides a third different proof of the filter normal form for a positive definite Hermitian matrix $A \in M_k \otimes M_k$, since $G_A : M_k \rightarrow M_k$ has that property in this case.

The authors of [15] adapted the Sinkhorn-Knopp algorithm in order to obtain the filter normal form for a positive definite Hermitian matrix $A \in M_k \otimes M_m$. Now, it can be shown that $G_A((\cdot)^t)$ is a completely positive map. Thus, they have implicitly used their iterative method for completely positive maps composed with transposition (the map G_A).

Here, we consider an adaptation of Sinkhorn-Knopp algorithm for any positive map $T : M_k \rightarrow M_m$. We define the notion of support and total support for positive maps. We show that a positive map $T : M_k \rightarrow M_m$ is equivalent to a doubly stochastic map if and only if $T(Id)$ and $T^*(Id)$ are invertible matrices and there are orthogonal projections $V_i \in M_k$, $W_i \in M_m$, $1 \leq i \leq s$, such that $\mathbb{C}^k = \bigoplus_{i=1}^s \Im(V_i)$, $\mathbb{C}^m = \bigoplus_{i=1}^s \Im(W_i)$ and, for every i , $T(V_i M_k V_i) \subset W_i M_m W_i$, $\frac{\text{rank}(W_i)}{\text{rank}(V_i)} = \frac{m}{k}$ and $\text{rank}(X)\text{rank}(W_i) < \text{rank}(T(X))\text{rank}(V_i)$ for every positive semidefinite Hermitian matrix $X \in V_i M_k V_i$ with $0 < \text{rank}(X) < \text{rank}(V_i)$. This consequently solves the conjecture 1.26 of [7].

Another useful result is the following: A positive map $T : M_k \rightarrow M_m$ is equivalent to a doubly stochastic map if and only if there are invertible matrices $X' \in M_k$, $Y' \in M_m$ such that $Y'T(X'XX'^*)Y'^*$ has total support. Thus, a necessary and sufficient condition for the existence of the filter normal form of a positive semidefinite Hermitian matrix $A \in M_k \otimes M_m$ is the existence of invertible matrices $X' \in M_k$, $Y' \in M_m$ such that $Y'G_A(X'XX'^*)Y'^*$ has total support.

The condition of total support for a positive map $T : M_k \rightarrow M_m$ cannot be easily checked, unless k and m are coprime, in this case support and total support are equivalent (See 2.5 and 2.10). However, there are some interesting sufficient conditions that grant the existence of the filter normal form. For example, if $A \in M_k \otimes M_m \simeq M_{km}$ and $\dim(\ker(A)) < k - 1$, if $k = m$, and $\dim(\ker(A)) < \min\{k, m\}$, if $k \neq m$. If $G_A(Id)$ and $G_A^*(Id)$ are invertible matrices and $\dim(\ker(A)) < \frac{\max\{k, m\}}{\min\{k, m\}}$.

This paper is organized as follows. In Section 1, we extend the definitions of support and total support to non-square matrices (definition 1.2). In Section 2, we define positive maps with support and total support (definition 2.2). We describe an adaptation of Sinkhorn and Knopp algorithm for positive maps $T : M_k \rightarrow M_m$ (algorithm 2.6) and we obtain a necessary and sufficient condition for the equivalence of a positive map with a doubly stochastic one (theorems 2.11 and 2.12). In Section 3, we address the filter normal form problem for a positive semidefinite Hermitian matrix $A \in M_k \otimes M_m$. As a consequence of our main theorems, we prove that the existence of this form is equivalent to the existence of invertible matrices $X' \in M_k$, $Y' \in M_m$ such that $Y'G_A(X'XX'^*)Y'^*$ has total support (theorem 3.1). We show that this last condition can be granted, if $k = m$ and $\dim(\ker(A)) < k - 1$, or if $k \neq m$ and $\dim(\ker(A)) < \min\{k, m\}$ or if $G_A(Id)$ and $G_A^*(Id)$ are invertible matrices and $\dim(\ker(A)) < \frac{\max\{k, m\}}{\min\{k, m\}}$ (theorems 3.3 and 3.4). We also present an example of a separable matrix which can not be put in the filter normal form (corollary 3.2).

We shall adopt the following notation.

Notation: Let $M_{k \times m}$ denote the set of complex matrix with k rows and m columns and $M_k = M_{k \times k}$. Denote by $\|A\|_2$ the spectral norm of the square matrix $A \in M_k$, $\Im(A)$ its image (range) and $\ker(A)$ its kernel. Denote by P_k the set of positive semidefinite Hermitian matrices of order k and VM_kV the set $\{VXV, X \in M_k\}$, where $V \in M_k$ is an orthogonal projection. Let A^\perp be the orthogonal projection onto $\ker(A)$, where $A \in M_k$. Let $(x_i)_{i=1}^k$ denote a column vector. If $x_i > 0$, for every i , then we shall say that $(x_i)_{i=1}^k$ is a positive vector. Let $A \odot B$ denote the Hadamard product (the coordinatewise product) and $A \otimes B$ the Kronecker product of the matrices A, B . We shall denote by $1_{m \times k}$ the matrix in $M_{m \times k}$ with all entries equal to 1.

Define $\sigma(A) = \prod_{i=1}^k a_{i\sigma(i)}$, where $A = (a_{ij})$ is a matrix of order k and σ a permutation of S_k . If $\emptyset \neq \alpha \subset \{1, \dots, k\}$ and $\emptyset \neq \beta \subset \{1, \dots, m\}$ then $A[\alpha|\beta]$ denotes the submatrix of $A \in M_{k \times m}$ using rows α and columns β , $A(\alpha|\beta)$ denotes the submatrix of A using rows and columns complementaries to α, β and $|\alpha|$ shall denote the cardinality of α . Let $A = \bigoplus_{i=1}^s A[\alpha_i|\beta_i] \in M_{k \times m}$ be such that $a_{ij} = 0$, if $(i, j) \notin \cup_{i=1}^s \alpha_i \times \beta_i$ and $\{1, \dots, k\} = \bigcup_{i=1}^s \alpha_i$, $\{1, \dots, m\} = \bigcup_{i=1}^s \beta_i$, where the sets α_i , $1 \leq i \leq s$, are disjoint and non empty, and the sets β_i , $1 \leq i \leq s$, are disjoint and non empty. This matrix shall be called the direct sum of $A[\alpha_i|\beta_i]$, $1 \leq i \leq s$. Let $\langle A, B \rangle = \text{tr}(AB^*)$ for $A, B \in M_k$.

1. A SLIGHT MODIFICATION OF SINKHORN-KNOPP IDEAS

The definitions of support and total support for square matrices play a very important role in Sinkhorn-Knopp theorem. In this section we extend these notions to non-square matrices and we adapt one key lemma (lemma 1.6), used by them in order to obtain their result, for non-square matrices. In the next section, we define positive maps with support and total support.

Definition 1.1. We say that $A = (a_{ij}) \in M_k$ has support, if there is a permutation $\sigma \in S_k$ such that $\sigma(A) \neq 0$. We say that A has total support, if for every $a_{i_0 j_0} \neq 0$, there is a permutation $\sigma \in S_k$ such that $\sigma(i_0) = j_0$ and $\sigma(A) = \prod_{i=1}^k a_{i\sigma(i)} \neq 0$, or equivalently, the matrix $A(\{i_0\}|\{j_0\})$ has support.

One way to extend the ideas of support and total support to non-square matrices is the following definition.

Definition 1.2. We say that $A = (a_{ij}) \in M_{k \times m}$ has support (total support), if $A \otimes 1_{m \times k} \in M_{km}$ has support (total support).

This extension is quite natural, since $A \in M_k$ has support (total support) if and only if $A \otimes 1_{k \times k} \in M_{k^2}$ has support (total support) by item (3) of lemma 1.5. In order to prove this lemma, we shall need the following result and a very simple corollary. The reader can find its proof in [11, pg 97].

Theorem 1.3. (Frobenius-König Theorem) The matrix $A \in M_k$ does not have support if and only if there is an identically zero submatrix $A[\alpha|\beta]$ such that $|\alpha| + |\beta| > k$.

A very simple corollary of this theorem is the following.

Corollary 1.4. $A \in M_k$ does not have total support if and only if there is an identically zero submatrix $A[\alpha|\beta]$ such that or $|\alpha| + |\beta| > k$ or $|\alpha| + |\beta| = k$ and $A(\alpha|\beta)$ is not identically zero.

The next lemma extend these two results to non-square matrices.

Lemma 1.5. Let $A \in M_{k \times m}$. Then

- (1) A does not have support if and only if there is an identically zero submatrix $A[\alpha|\beta]$ such that $|\alpha|m + |\beta|k > km$.
- (2) A does not have total support if and only if there is an identically zero submatrix $A[\alpha|\beta]$ such that or $|\alpha|m + |\beta|k > km$ or $|\alpha|m + |\beta|k = km$ and $A(\alpha|\beta)$ is not identically zero.
- (3) If $k = m$ then A has support (total support) if and only if $A \otimes 1_{k \times k}$ has support (total support).
- (4) If k and m are coprime then A has support if and only if A has total support.
- (5) If $k \neq m$ and the cardinality of $\{(i, j) | a_{ij} = 0\} < \min\{k, m\}$ then A has total support.
- (6) If $k = m$ and the cardinality of $\{(i, j) | a_{ij} = 0\} < k - 1$ then A has total support.
- (7) If A has no column or row identically zero and the cardinality of $\{(i, j) | a_{ij} = 0\} < \frac{\max\{k, m\}}{\min\{k, m\}}$ then A has total support.
- (8) If $A = \bigoplus_{i=1}^s A[\alpha_i|\beta_i]$, $\frac{|\beta_i|}{|\alpha_i|} = \frac{m}{k}$ and $A[\alpha_i|\beta_i]$ has support (total support) ($1 \leq i \leq s$) then A has support (total support).

Proof. Let $C = A \otimes 1_{m \times k}$. Notice that any identically zero $C[\alpha'|\beta']$ is a submatrix of some identically zero $A[\alpha|\beta] \otimes 1_{m \times k}$. Since any $A[\alpha|\beta] \otimes 1_{m \times k}$ is also a submatrix of C then the identically zero matrices $C[\alpha'|\beta']$ with maximum $|\alpha'| + |\beta'|$ are the identically zero matrices $A[\alpha|\beta] \otimes 1_{m \times k}$ with maximum $|\alpha|m + |\beta|k$.

(1) $C = A \otimes 1_{m \times k}$ has no support if and only if there is an identically zero submatrix $C[\alpha'|\beta'] = A[\alpha|\beta] \otimes 1_{m \times k}$ such that $|\alpha'| + |\beta'| = |\alpha|m + |\beta|k > km$, by theorem 1.3.

(2) $C = A \otimes 1_{m \times k}$ does not have total support if and only if there is an identically zero submatrix $C[\alpha'|\beta'] = A[\alpha|\beta] \otimes 1_{m \times k}$ such that or $|\alpha'| + |\beta'| = |\alpha|m + |\beta|k > km$ or $|\alpha'| + |\beta'| = |\alpha|m + |\beta|k = km$ and $C(\alpha'|\beta') = A(\alpha|\beta) \otimes 1_{m \times k}$ is not identically zero, by corollary 1.4.

(3) Let $k = m$ in the proofs of items (1), (2) and then use theorem 1.3 and corollary 1.4.

(4) If k and m are coprime then there are no positive integers x, y such that $xm + yk = mk$. Now, use (1) and (2).

(5), (6) Let $C[\alpha'|\beta']$ be identically zero with maximum $|\alpha'| + |\beta'|$ then $C[\alpha'|\beta'] = A[\alpha|\beta] \otimes 1_{m \times k}$. Thus, $|\alpha'| + |\beta'| = |\alpha|m + |\beta|k = \frac{|\alpha|m + |\beta|k}{|\alpha| + |\beta|}(|\alpha| + |\beta|) \leq \frac{|\alpha|m + |\beta|k}{|\alpha| + |\beta|}(|\alpha||\beta| + 1)$.

Since $A[\alpha|\beta]$ is identically zero then $|\alpha||\beta|$ is smaller than $\min\{k, m\}$, for $k \neq m$, or smaller than $k - 1$, if $k = m$, by hypothesis. Therefore, if $k \neq m$ then $\frac{|\alpha|m + |\beta|k}{|\alpha| + |\beta|}(|\alpha||\beta| + 1) < \max\{m, k\} \times \min\{km\} = km$. If $k = m$ then $\frac{|\alpha|m + |\beta|k}{|\alpha| + |\beta|}(|\alpha||\beta| + 1) \leq k(k - 1) < k^2$. The results follow by (2).

(7) Let $A[\alpha|\beta]$ be identically zero. Since there are no rows or columns identically zero then $|\alpha| \leq k - 1$ and $|\beta| \leq m - 1$.

Next, if $|\alpha|m + |\beta|k \geq km$ then $(k - 1)m + |\beta|k \geq km$ and $|\alpha|m + (m - 1)k \geq km$. Thus, $|\beta| \geq \frac{m}{k}$ and $|\alpha| \geq \frac{k}{m}$, which is impossible, since the cardinality of $\{(i, j) | a_{ij} = 0\} < \frac{\max\{k, m\}}{\min\{k, m\}}$. Therefore, $|\alpha|m + |\beta|k < km$ and A has total support by item (2).

(8) Notice that $A \otimes 1_{m \times k} = \bigoplus_{i=1}^s (A[\alpha_i|\beta_i] \otimes 1_{m \times k})$ and $|\alpha_i|m = |\beta_i|k$ for every i . So $A \otimes 1_{m \times k}$ is a direct sum of square matrices with support (total support) then A has support (total support). \square

The next lemma is a slight modification of lemma 2 in [14].

Lemma 1.6. *Let $(v_n)_{n \in \mathbb{N}}$ be a sequence of positive vectors of \mathbb{R}^k and $(w_n)_{n \in \mathbb{N}}$ a sequence of positive vectors of \mathbb{R}^m . Let $A = (a_{ij}) \in M_{k \times m}$ be a matrix with total support. If for every $a_{ij} \neq 0$ the corresponding entry of $v_n w_n^t$ (i.e. $(v_n w_n^t)_{i,j} = v_{i,n} w_{j,n}$) converges to a positive limit then there are two sequences $(v'_n)_{n \in \mathbb{N}}$, $(w'_n)_{n \in \mathbb{N}}$ of positive vectors of \mathbb{R}^k and \mathbb{R}^m converging to positive vectors v, w , respectively, such that $v_n w_n^t = v'_n w_n'^t$ for every n (i.e. $v_{i,n} w_{j,n} = v'_{i,n} w'_{j,n}$, for every i, j, n).*

Proof. Let us assume $k \neq m$, since the case $k = m$ is lemma 2 of [14].

By definition 1.2, the square matrix $A \otimes 1_{m \times k}$ has total support.

Notice that an entry $v_{i,n} w_{j,n}$ of the matrix $v_n w_n^t \otimes 1_{m \times k}$ corresponds to an entry a_{ij} of the matrix $A \otimes 1_{m \times k}$. Thus, whenever an entry of the matrix $A \otimes 1_{m \times k}$ is not zero, the corresponding entry of the rank 1 matrix $v_n w_n^t \otimes 1_{m \times k} = (v_n \otimes 1_{m \times 1})(w_n^t \otimes 1_{1 \times k})$ converges to a positive number.

Since $A \otimes 1_{m \times k} \in M_{km \times km}$ is square with total support and $v_n \otimes 1_{m \times 1}, w_n \otimes 1_{k \times 1} \in \mathbb{R}^{km}$ are positive vectors then, by the case $k = m$, there are two sequences $(v'_n)_{n \in \mathbb{N}}$, $(w'_n)_{n \in \mathbb{N}}$ of positive vectors of \mathbb{R}^{km} converging to positive vectors v, w , such that $(v_n \otimes 1_{m \times 1})(w_n^t \otimes 1_{1 \times k}) = v'_n w_n'^t$ for

every n . Thus, there are subvectors $v''_n \in \mathbb{R}^k$ of v'_n and $w''_n \in \mathbb{R}^m$ of w'_n converging to positive vectors $v'' \in \mathbb{R}^k$, $w'' \in \mathbb{R}^m$, respectively, such that $v_n w_n^t = v''_n w''_n{}^t$, for every n . \square

The next lemma shall be used later. Its proof is left to the reader.

Lemma 1.7. (1) If A, B are matrices of order k then $\sigma(A \odot B) = \sigma(A)\sigma(B)$, for any σ .
 (2) If $v = (v_i)_{i=1}^k$, $w = (w_i)_{i=1}^k$ then $\sigma(vw^t) = \prod_{i=1}^k v_i w_i$, for any σ .
 (3) If $v = (v_i)_{i=1}^k$, $r = (r_i)_{i=1}^m$ then $\sigma(vr^t \otimes 1_{m \times k}) = (\prod_{i=1}^k v_i)^m (\prod_{i=1}^m w_i)^k$, for any $\sigma \in S_{mk}$.

2. SINKHORN-KNOPP ALGORITHM FOR POSITIVE MAPS

In this section, we discuss an adaptation of Sinkhorn-Knopp algorithm for positive maps. A similar adaptation was used in [15], in order to obtain the so-called filter normal form for positive definite states.

Here, we describe an algorithm for general positive maps $T : M_k \rightarrow M_m$. We define positive maps with support and total support (definition 2.2) and we show that if $T : M_k \rightarrow M_m$ has support then the limit points of the sequence produced by the algorithm are doubly stochastic. We also show that if $T : M_k \rightarrow M_m$ has total support then there are invertible matrices $X' \in M_k$, $Y' \in M_m$ such that $Y'T(X'XX'^*)Y'^*$ is doubly stochastic (lemma 2.9). Differently from the matrix case, the condition of total support is not necessary for the equivalence of a positive map with a doubly stochastic one (See remark 2.13).

As a consequence of lemma 2.9, we obtain a necessary and sufficient condition for the equivalence of a positive map with a doubly stochastic map (theorems 2.11 and 2.12).

In the next section, we discuss the filter normal form for states that are not positive definite and we provide easy sufficient conditions for the existence of this form.

Definition 2.1. Let $T : M_k \rightarrow M_m$ be a positive map. We say that T is doubly stochastic if $T(\frac{Id}{\sqrt{k}}) = \frac{Id}{\sqrt{m}}$ and $T^*(\frac{Id}{\sqrt{m}}) = \frac{Id}{\sqrt{k}}$, where T^* is the adjoint of T with respect to the trace inner product.

The interested reader can find more information concerning doubly stochastic maps in [4, 9, 12].

Definition 2.2. We say that $T : M_k \rightarrow M_m$ has support (total support), if for every orthonormal bases $\{v_1, \dots, v_k\}$ of \mathbb{C}^k and $\{w_1, \dots, w_m\}$ of \mathbb{C}^m , the matrix $(tr(T(v_i \bar{v}_i^t) w_j \bar{w}_j^t)) \in M_{k \times m}$ has support (total support). More generally, we say that $T : VM_k V \rightarrow WM_m W$ has support (total support), where $V \in M_k$ and $W \in M_m$ are orthogonal projections, if for every orthonormal bases $\{v_1, \dots, v_{k'}\}$ of $\mathfrak{S}(V)$ and $\{w_1, \dots, w_{m'}\}$ of $\mathfrak{S}(W)$, the matrix $(tr(T(v_i \bar{v}_i^t) w_j \bar{w}_j^t)) \in M_{k' \times m'}$ has support (total support).

The next lemma provides another description of positive maps $T : M_k \rightarrow M_m$ with support and total support. A similar description is valid for positive maps $T : VM_k V \rightarrow WM_m W$ (See remark 2.4).

Lemma 2.3. Let $T : M_k \rightarrow M_m$ be a positive map. Then

- (1) $T : M_k \rightarrow M_m$ has support if and only if for every $A \in P_k$, $\text{rank}(A)m \leq \text{rank}(T(A))k$.
- (2) $T : M_k \rightarrow M_m$ has total support if and only if for every $A \in P_k$, $\text{rank}(A)m < \text{rank}(T(A))k$ or $\text{rank}(A)m = \text{rank}(T(A))k$ and $\mathfrak{S}(T(A)) = \mathfrak{S}(T(A)^\perp)^\perp$.

Proof. (1) Notice that if $A \in P_k$ and $U \in P_k$ is the orthogonal projection onto $\mathfrak{S}(A)$ then $\mathfrak{S}(T(A)) = \mathfrak{S}(T(U))$, since $T : M_k \rightarrow M_m$ is a positive map. Thus, we just need to prove the inequality for orthogonal projections.

Now, T has no support if and only if there are orthonormal bases $\{v_1, \dots, v_k\} \subset \mathbb{C}^k$ and $\{w_1, \dots, w_m\} \subset \mathbb{C}^m$ such that the matrix $A = (tr(T(v_i \bar{v}_i^t) w_j \bar{w}_j^t)) \in M_{k \times m}$ has no support. By

lemma 1.5, this is equivalent to the existence of an identically zero submatrix $A[\alpha|\beta]$ such that $|\alpha|m + |\beta|k > km$. Define $V = \sum_{i \in \alpha} v_i \bar{v}_i^t$ and $W = \sum_{j \in \beta} w_j \bar{w}_j^t$. Notice that $A[\alpha|\beta]$ is identically zero if and only if $\text{tr}(T(V)W) = 0$.

Next, the existence of orthogonal projections $V \in M_k, W \in M_m$ such that $\text{tr}(T(V)W) = 0$ and $\text{rank}(V)m + \text{rank}(W)k > km$ is equivalent to $\text{rank}(V)m + \dim(\ker(T(V)))k > km$, since $\Im(W) \subset \ker(T(V))$. Finally, since $\text{rank}(V)m > (m - \dim(\ker(T(V))))k = \text{rank}(T(V))k$ then T has no support if and only if there is an orthogonal projection $V \in M_k$ such that $\text{rank}(V)m > \text{rank}(T(V))k$.

(2) First, let us assume that T has total support. Therefore T has support and $m(\text{rank}(A)) \leq k(\text{rank}(T(A)))$, for every $A \in P_k$, by item (1). Notice that, if $\text{rank}(A) \in \{0, k\}$ then $\Im(T(A)) = \Im(T(A^\perp)^\perp)$. Let $m(\text{rank}(A)) = k(\text{rank}(T(A)))$, for some $A \in P_k$ with $0 < \text{rank}(A) < k$. Thus, $0 < \text{rank}(T(A)) < m$.

Let $\{v_1, \dots, v_k\} \subset \mathbb{C}^k$ be an orthonormal basis of eigenvectors of A and $\{w_1, \dots, w_m\} \subset \mathbb{C}^m$ be an orthonormal basis of eigenvectors of $T(A)$. Thus, $A = \sum_{i=1}^k a_i v_i \bar{v}_i^t$ and $T(A) = \sum_{j=1}^m b_j w_j \bar{w}_j^t$, where a_i, b_j are the non-negative eigenvalues. Let $\alpha = \{i | a_i > 0\}$ and $\beta = \{j | b_j > 0\}$.

Consider the matrix $C = (\text{tr}(T(v_i \bar{v}_i^t) w_j \bar{w}_j^t)) \in M_{k \times m}$. Notice that $C[\alpha|\beta]$ is identically zero. Since C has total support and $|\alpha|m + |\beta|k = \text{rank}(A)m + (m - \text{rank}(T(A)))k = mk$ then $C(\alpha|\beta)$ must be identically zero, by item (2) of lemma 1.5. Hence, $\text{tr}(T(A^\perp)T(A)) = 0$. So $\Im(T(A)) \subset \ker(T(A^\perp)) = \Im(T(A^\perp)^\perp)$.

Next, $\text{rank}(T(A^\perp)^\perp)k = (m - \text{rank}(T(A^\perp)))k \leq mk - \text{rank}(A^\perp)m = \text{rank}(A)m \leq \text{rank}(T(A))k$. Thus, $\Im(T(A)) = \Im(T(A^\perp)^\perp)$.

For the converse, let $\{r_1, \dots, r_k\} \subset \mathbb{C}^k$ and $\{s_1, \dots, s_m\} \subset \mathbb{C}^m$ be any orthonormal bases. Consider the matrix $B = (\text{tr}(T(r_i \bar{r}_i^t) s_j \bar{s}_j^t)) \in M_{k \times m}$.

If $B[\alpha'|\beta']$ is identically zero then $\text{tr}(T(R)S) = 0$, where $R = \sum_{i \in \alpha'} r_i \bar{r}_i^t$ and $S = \sum_{j \in \beta'} s_j \bar{s}_j^t$. Thus, $\Im(T(R)) \subset \ker(S)$ which implies $\text{rank}(T(R))k \leq (m - \text{rank}(S))k = mk - |\beta'|k$. By assumption, $\text{rank}(R)m \leq \text{rank}(T(R))k$ then $|\alpha'|m = \text{rank}(R)m \leq \text{rank}(T(R))k \leq mk - |\beta'|k$. Hence, $|\alpha'|m + |\beta'|k \leq km$.

Now, assume $|\alpha'|m = km - |\beta'|k$. Since $|\alpha'|m = \text{rank}(R)m \leq \text{rank}(T(R))k \leq (m - \text{rank}(S))k = mk - |\beta'|k$ then $\text{rank}(R)m = \text{rank}(T(R))k = (m - \text{rank}(S))k$.

Next, by hypothesis, $\Im(T(R)) = \Im(T(R^\perp)^\perp)$. Since $\Im(T(R)) \subset \ker(S) = \Im(S^\perp)$ and $\text{rank}(T(R)) = m - \text{rank}(S) = \text{rank}(S^\perp)$ then $\Im(S^\perp) = \Im(T(R))$. Thus, $\Im(S^\perp) = \Im(T(R)) = \Im(T(R^\perp)^\perp)$. Therefore, $\text{tr}(T(R^\perp)S^\perp) = 0$ which is equivalent to $B(\alpha'|\beta')$ being identically zero. So, by item (2) of lemma 1.5, B has total support. Thus, T has total support. \square

Remark 2.4. Notice that if $T : VM_kV \rightarrow WM_mW$ is a positive map such that $\Im(T(V)) = \Im(W)$ and $\text{rank}(A)\text{rank}(W) \leq \text{rank}(T(A))\text{rank}(V)$, for every $A \in VM_kV \cap P_k$, then T has support. Moreover, if $\text{rank}(A)\text{rank}(W) < \text{rank}(T(A))\text{rank}(V)$, for every $A \in VM_kV \cap P_k$ with $0 < \text{rank}(A) < \text{rank}(V)$, then T has total support.

Examples 2.5. Let $T : M_k \rightarrow M_m$ be a positive map, $S : M_k \rightarrow M_m$ a doubly stochastic map, $\{v_1, \dots, v_k\}$ any orthonormal basis of \mathbb{C}^k and $\{w_1, \dots, w_m\}$ any orthonormal basis of \mathbb{C}^m .

- (1) S has total support, since $(\text{tr}(S(v_i \bar{v}_i^t) w_j \bar{w}_j^t))_{k \times m} \otimes 1_{m \times k}$ is a doubly stochastic matrix scaled by \sqrt{km} and every doubly stochastic matrix has total support (See [14]).
- (2) If k and m are coprime then $T : M_k \rightarrow M_m$ has total support iff T has support, by item (4) of lemma 1.5.

Let us describe an adaptation of Sinkhorn and Knopp algorithm for positive maps.

Algorithm 2.6. Let $T : M_k \rightarrow M_m$ be a positive map such that $T(\text{Id})$ and $T^*(\text{Id})$ are positive definite Hermitian matrices. Define

$$\begin{aligned}
X_0 &= Id \in M_k, Y_0 = (\frac{Id}{\sqrt{m}})^{\frac{1}{2}} T(X_0 \frac{Id}{\sqrt{k}} X_0^*)^{-\frac{1}{2}}, \\
A_0 &= X_0^* T^*(Y_0^* \frac{Id}{\sqrt{m}} Y_0) X_0, \\
X_1 &= X_0 A_0^{-\frac{1}{2}} (\frac{Id}{\sqrt{k}})^{\frac{1}{2}}, \\
B_0 &= Y_0 T(X_1 \frac{Id}{\sqrt{k}} X_1^*) Y_0^*
\end{aligned}$$

Notice that A_0 is a positive definite Hermitian matrix, since $Y_0^* Y_0$ and $T^*(Id)$ are positive definite Hermitian matrices. Analogously, B_0 is a positive definite Hermitian matrix. Notice also that X_0, Y_0, X_1 are invertible matrices.

Supposed defined $X_n, Y_n, A_n, X_{n+1}, B_n$ such that A_n, B_n are positive definite Hermitian matrices and X_n, Y_n, X_{n+1} are invertible matrices. Define

$$\begin{aligned}
Y_{n+1} &= (\frac{Id}{\sqrt{m}})^{\frac{1}{2}} B_n^{-\frac{1}{2}} Y_n, \\
A_{n+1} &= X_{n+1}^* T^*(Y_{n+1}^* \frac{Id}{\sqrt{m}} Y_{n+1}) X_{n+1}, \\
X_{n+2} &= X_{n+1} A_{n+1}^{-\frac{1}{2}} (\frac{Id}{\sqrt{k}})^{\frac{1}{2}}, \\
B_{n+1} &= Y_{n+1} T(X_{n+2} \frac{Id}{\sqrt{k}} X_{n+2}^*) Y_{n+1}^*.
\end{aligned}$$

Notice that A_{n+1} is a positive definite Hermitian matrix, since $Y_{n+1}^* Y_{n+1}$ and $T^*(Id)$ are positive definite Hermitian matrices. Analogously, B_{n+1} is a positive definite Hermitian matrix. Notice also that $X_{n+1}, Y_{n+1}, X_{n+2}$ are invertible matrices.

Lemma 2.7. Let $T : M_k \rightarrow M_m$ be a positive map such that $T(Id), T^*(Id)$ are positive definite Hermitian matrices. Let X_n, A_n, Y_n, B_n be as defined in algorithm 2.6. Then,

- (1) $Y_n T(X_n \frac{Id}{\sqrt{k}} X_n^*) Y_n^* = \frac{Id}{\sqrt{m}}, X_{n+1}^* T^*(Y_n^* \frac{Id}{\sqrt{m}} Y_n) X_{n+1} = \frac{Id}{\sqrt{k}}, n > 0$
- (2) $tr(A_n) = \sqrt{k}, tr(B_n) = \sqrt{m}, n > 0$
- (3) $0 < \det(X_n \otimes Y_n) \leq \det(X_{n+1} \otimes Y_{n+1})$
- (4) If $(\det(X_n \otimes Y_n))_{n \in \mathbb{N}}$ is bounded then every limit point of $(Y_n T(X_n X X_n^*) Y_n^*)_{n \in \mathbb{N}}$ is a doubly stochastic map.

Proof. (1) Notice that $Y_{n+1} T(X_{n+1} \frac{Id}{\sqrt{k}} X_{n+1}^*) Y_{n+1}^* = (\frac{Id}{\sqrt{m}})^{\frac{1}{2}} B_n^{-\frac{1}{2}} Y_n T(X_{n+1} \frac{Id}{\sqrt{k}} X_{n+1}^*) Y_n^* B_n^{-\frac{1}{2}} (\frac{Id}{\sqrt{m}})^{\frac{1}{2}} = (\frac{Id}{\sqrt{m}})^{\frac{1}{2}} B_n^{-\frac{1}{2}} B_n B_n^{-\frac{1}{2}} (\frac{Id}{\sqrt{m}})^{\frac{1}{2}} = \frac{Id}{\sqrt{m}}$. Next, $X_{n+1}^* T^*(Y_n^* \frac{Id}{\sqrt{m}} Y_n) X_{n+1} = (\frac{Id}{\sqrt{k}})^{\frac{1}{2}} A_n^{-\frac{1}{2}} X_n^* T^*(Y_n^* \frac{Id}{\sqrt{m}} Y_n) X_n A_n^{-\frac{1}{2}} (\frac{Id}{\sqrt{k}})^{\frac{1}{2}} = (\frac{Id}{\sqrt{k}})^{\frac{1}{2}} A_n^{-\frac{1}{2}} A_n A_n^{-\frac{1}{2}} (\frac{Id}{\sqrt{k}})^{\frac{1}{2}} = \frac{Id}{\sqrt{k}}$.

(2) Now, $tr(A_n) = \sqrt{k} \langle X_n^* T^*(Y_n^* \frac{Id}{\sqrt{m}} Y_n) X_n, \frac{Id}{\sqrt{k}} \rangle = \sqrt{k} \langle \frac{Id}{\sqrt{m}}, Y_n T(X_n \frac{Id}{\sqrt{k}} X_n^*) Y_n^* \rangle = \sqrt{k} \langle \frac{Id}{\sqrt{m}}, \frac{Id}{\sqrt{m}} \rangle = \sqrt{k}$ and $tr(B_n) = \sqrt{m} \langle Y_n T(X_{n+1} \frac{Id}{\sqrt{k}} X_{n+1}^*) Y_n^*, \frac{Id}{\sqrt{m}} \rangle = \sqrt{m} \langle \frac{Id}{\sqrt{k}}, X_{n+1}^* T^*(Y_n^* \frac{Id}{\sqrt{m}} Y_n) X_{n+1} \rangle = \sqrt{m} \langle \frac{Id}{\sqrt{k}}, \frac{Id}{\sqrt{k}} \rangle = \sqrt{m}$.

(3) Since $(\sqrt{k} A_n) \otimes (\sqrt{m} B_n)$ is a positive definite hermitian matrix then $\det((\sqrt{k} A_n) \otimes (\sqrt{m} B_n)) \leq \left(\frac{tr((\sqrt{k} A_n) \otimes (\sqrt{m} B_n))}{km} \right)^{km} = 1$. Thus, $\frac{\det(X_n \otimes Y_n)}{\det(X_{n+1} \otimes Y_{n+1})} = \det(X_{n+1}^{-1} X_n \otimes Y_n Y_{n+1}^{-1}) = \det((\sqrt{k} A_n)^{\frac{1}{2}} \otimes (\sqrt{m} B_n)^{\frac{1}{2}}) = \det((\sqrt{k} A_n) \otimes (\sqrt{m} B_n))^{\frac{1}{2}} \leq 1$. Therefore, $0 < \det(X_1 \otimes Y_1) \leq \det(X_n \otimes Y_n) \leq \det(X_{n+1} \otimes Y_{n+1})$.

(4) Since $(\det(X_n \otimes Y_n))_{n \in \mathbb{N}}$ is bounded then, by item (3), $0 < L = \lim_{n \rightarrow \infty} \det(X_n \otimes Y_n)$.

Thus, $1 = \lim_{n \rightarrow \infty} \left(\frac{\det(X_n \otimes Y_n)}{\det(X_{n+1} \otimes Y_{n+1})} \right)^2 = \lim_{n \rightarrow \infty} \det(X_{n+1}^{-1} X_n \otimes Y_n Y_{n+1}^{-1})^2 = \lim_{n \rightarrow \infty} \det((\sqrt{k} A_n) \otimes (\sqrt{m} B_n))$.

Let C be a limit point of the sequence $((\sqrt{k} A_n) \otimes (\sqrt{m} B_n))_{n \in \mathbb{N}}$ (there are limit points since $tr((\sqrt{k} A_n) \otimes (\sqrt{m} B_n)) = km$ and A_n, B_n are positive definite) then $\det(C) = 1$ and $tr(C) = km$,

since C is positive semidefinite then $C = Id \otimes Id$. Hence, $\lim_{n \rightarrow \infty} (\sqrt{k}A_n) \otimes (\sqrt{m}B_n) = Id \otimes Id$ and $\lim_{n \rightarrow \infty} (\sqrt{k}A_n)tr(\sqrt{m}B_n) = \lim_{n \rightarrow \infty} (\sqrt{k}A_n)m = mId$. So, $\lim_{n \rightarrow \infty} A_n = \frac{Id}{\sqrt{k}}$.

Since the operator norm of a positive map induced by the spectral norm is attained at the identity ([2, corollary 2.3.8]) then $\|Y_n T(X_n X X_n^*) Y_n^*\| = \|\sqrt{k} Y_n T(X_n \frac{Id}{\sqrt{k}} X_n^*) Y_n^*\|_2 = \|\frac{\sqrt{k} Id}{\sqrt{m}}\|_2 = \frac{\sqrt{k}}{\sqrt{m}}$. Thus, there are limit points of the sequence of positive maps $(Y_n T(X_n X X_n^*) Y_n^*)_{n \in \mathbb{N}}$. Since $Y_n T(X_n \frac{Id}{\sqrt{k}} X_n^*) Y_n^* = \frac{Id}{\sqrt{m}}$ and $X_n^* T^*(Y_n^* \frac{Id}{\sqrt{m}} Y_n) X_n = A_n \xrightarrow{n \rightarrow \infty} \frac{Id}{\sqrt{k}}$ then these limit points are doubly stochastic. \square

Lemma 2.8. *Let $T : M_k \rightarrow M_m$ be a positive map such that $T(Id)$, $T^*(Id)$ are positive definite Hermitian matrices. Let $X_n \in M_k, Y_n \in M_m$ be the matrices defined in algorithm 2.6. If there are orthogonal projections $V_i \in M_k, W_i \in M_m, 1 \leq i \leq s$, such that $V_i V_j = 0, W_i W_j = 0$, for $i \neq j$, and $Id = \sum_{i=1}^s V_i, Id = \sum_{i=1}^s W_i$ and $T(V_i M_k V_i) \subset W_i M_m W_i$, for $1 \leq i \leq s$, then $X_n V_i = V_i X_n$ and $Y_n W_i = W_i Y_n$, for every i, n .*

Proof. First, notice that $\mathfrak{S}(T(V_i)) = \mathfrak{S}(W_i)$ for every i , since $T(Id)$ is positive definite and $T(V_i M_k V_i) \subset W_i M_m W_i$ for every i . Now, $\langle T^*(W_j), \sum_{i \neq j} V_i \rangle = \langle W_j, \sum_{i \neq j} T(V_i) \rangle = 0$. Thus, $\mathfrak{S}(T^*(W_j)) = \mathfrak{S}(V_j)$ and $T^*(W_j M_m W_j) \subset V_j M_k V_j$, for every j .

Since $X_0 = Id$ then $V_i X_0 = X_0 V_i$ for every i . Since $T(\frac{Id}{\sqrt{k}}) = \sum_{i=1}^s \frac{1}{\sqrt{k}} T(V_i)$ and $T(V_i) \in W_i M_m W_i$ then $T(\frac{Id}{\sqrt{k}}) W_i = W_i T(\frac{Id}{\sqrt{k}})$, for every i . Now, $Y_0 = (\frac{Id}{\sqrt{m}})^{\frac{1}{2}} T(\frac{Id}{\sqrt{k}})^{-\frac{1}{2}}$ is a polynomial of $T(\frac{Id}{\sqrt{k}})$, since $T(\frac{Id}{\sqrt{k}})$ is a positive definite Hermitian matrix. Therefore, $Y_0 W_i = W_i Y_0$, for every i .

Let us assume by induction that $X_n V_i = V_i X_n$ and $Y_n W_i = W_i Y_n$, for every i . Notice that $Y_n^* W_i = W_i Y_n^*$ for every i . Therefore, $Y_n^* Y_n \in \bigoplus_{i=1}^s W_i M_m W_i$.

Next, since X_n and V_i commute and $T^*(Y_n^* \frac{Id}{\sqrt{m}} Y_n) \in \bigoplus_{i=1}^s V_i M_k V_i$ then $A_n = X_n^* T^*(Y_n^* \frac{Id}{\sqrt{m}} Y_n) X_n$ commutes with V_i , for every i . Since A_n is a positive definite Hermitian matrix then $A_n^{-\frac{1}{2}}$ is a polynomial of A_n . Hence, V_i commutes with $X_{n+1} = X_n A_n^{-\frac{1}{2}} (\frac{Id}{\sqrt{k}})^{\frac{1}{2}}$ and X_{n+1}^* , for every i .

Therefore, $X_{n+1} X_{n+1}^* \in \bigoplus_{i=1}^s V_i M_k V_i$ and $T(X_{n+1} X_{n+1}^*) \in \bigoplus_{i=1}^s W_i M_m W_i$.

Since Y_n and W_i commute and $T(X_{n+1} X_{n+1}^*) \in \bigoplus_{i=1}^s W_i M_m W_i$ then $B_n = Y_n T(X_{n+1} \frac{Id}{\sqrt{k}} X_{n+1}^*) Y_n^*$ commutes with W_i , for every i . Since B_n is a positive definite Hermitian matrix then $B_n^{-\frac{1}{2}}$ is a polynomial of B_n . Therefore, W_i commutes with $Y_{n+1} = (\frac{Id}{\sqrt{m}})^{\frac{1}{2}} B_n^{-\frac{1}{2}} Y_n$, for every i . The induction is complete. \square

Lemma 2.9. *Let $T : M_k \rightarrow M_m$ be a positive map such that $T(Id)$, $T^*(Id)$ are positive definite Hermitian matrices. Let $T_1 : M_k \rightarrow M_m$ be any limit point of the sequence of positive maps $(Y_n T(X_n X X_n^*) Y_n^*)_{n \in \mathbb{N}}$, where $X_n \in M_k, Y_n \in M_m$ are defined in algorithm 2.6. Let $V_i \in M_k, W_i \in M_m, 1 \leq i \leq s$, be orthogonal projections such that $V_i V_j = 0, W_i W_j = 0$, for $i \neq j$, and $Id = \sum_{i=1}^s V_i, Id = \sum_{i=1}^s W_i$ and $T(V_i M_k V_i) \subset W_i M_m W_i$, for $1 \leq i \leq s$. Let $\text{rank}(W_i) = m_i$ and $\text{rank}(V_i) = k_i$.*

- (1) *If $T : V_i M_k V_i \rightarrow W_i M_m W_i$ has support and $\frac{m_i}{k_i} = \frac{m}{k}$, for every i , then T_1 is doubly stochastic.*
- (2) *If $T : V_i M_k V_i \rightarrow W_i M_m W_i$ has total support and $\frac{m_i}{k_i} = \frac{m}{k}$, for every i , then there are invertible matrices $X' \in M_k, Y' \in M_m$ such that $T_1(X) = Y' T(X' X X'^*) Y'^*$.*

Proof. Let $X_n = L_n D_n M_n^*, Y_n = S_n \tilde{D}_n R_n^*$, where $L_n, M_n \in M_k, S_n, R_n \in M_m$ are unitary matrices, and $D_n = \text{diag}(x_{1,n}, \dots, x_{k,n}), \tilde{D}_n = \text{diag}(y_{1,n}, \dots, y_{m,n})$ are positive diagonal matrices (i.e., SVD decompositions of X_n and Y_n).

By lemma 2.8, $X_n V_a = V_a X_n$ for $1 \leq a \leq s$. We can assume without loss of generality that the columns $k_1 + \dots + k_{a-1} + 1, \dots, k_1 + \dots + k_a$ of L_n form an orthonormal basis of $\mathfrak{S}(V_a)$ for

$1 \leq a \leq s$. Analogously, since $Y_n W_a = W_a Y_n$ (lemma 2.8), we can assume that the columns $m_1 + \dots + m_{a-1} + 1, \dots, m_1 + \dots + m_a$ of R_n form an orthonormal basis of $\mathfrak{S}(W_a)$ for $1 \leq a \leq s$.

Since the set of unitary matrices is compact, we can pass to a subsequence to ensure the convergence of L_n, M_n, S_n, R_n to unitary matrices L, M, S, R , respectively. In order to simplify our notation, we shall write $\lim_{n \rightarrow \infty} L_n = L$, $\lim_{n \rightarrow \infty} M_n = M$, $\lim_{n \rightarrow \infty} S_n = S$, $\lim_{n \rightarrow \infty} R_n = R$, $\lim_{n \rightarrow \infty} Y_n T(X_n X_n^*) Y_n^* = T_1(X)$.

Next, since $\lim_{n \rightarrow \infty} L_n = L$ and $\lim_{n \rightarrow \infty} R_n = R$ then the columns $k_1 + \dots + k_{a-1} + 1, \dots, k_1 + \dots + k_a$ of L form an orthonormal basis of $\mathfrak{S}(V_a)$ and the columns $m_1 + \dots + m_{a-1} + 1, \dots, m_1 + \dots + m_a$ of R form an orthonormal basis of $\mathfrak{S}(W_a)$ for $1 \leq a \leq s$.

Let $l_{i,n}, m_{i,n}, s_{i,n}, r_{i,n}, l_i, m_i, s_i, r_i$ be the columns i of $L_n, M_n, S_n, R_n, L, M, S, R$, respectively.

Consider the following matrices of order mk with non-negative entries:

$$C_n = (\text{tr}(Y_n T(X_n m_{i,n} \bar{m}_{i,n}^t X_n^*) Y_n^* s_{j,n} \bar{s}_{j,n}^t))_{k \times m} \otimes 1_{m \times k}, \quad A_n = (x_{i,n}^2 y_{j,n}^2)_{k \times m} \otimes 1_{m \times k} \text{ and} \\ B_n = (\text{tr}(T(l_{i,n} \bar{l}_{i,n}^t) r_{j,n} \bar{r}_{j,n}^t))_{k \times m} \otimes 1_{m \times k}.$$

Since $x_{i,n}^2 y_{j,n}^2 \text{tr}(T(l_{i,n} \bar{l}_{i,n}^t) r_{j,n} \bar{r}_{j,n}^t) = \text{tr}(Y_n T(X_n m_{i,n} \bar{m}_{i,n}^t X_n^*) Y_n^* s_{j,n} \bar{s}_{j,n}^t)$ then $C_n = A_n \odot B_n$.

Notice that $\lim_{n \rightarrow \infty} C_n = C$, $\lim_{n \rightarrow \infty} B_n = B$, where $C = (\text{tr}(T_1(m_i \bar{m}_i^t) s_j \bar{s}_j^t))_{k \times m} \otimes 1_{m \times k}$ and $B = (\text{tr}(T(l_i \bar{l}_i^t) r_j \bar{r}_j^t))_{k \times m} \otimes 1_{m \times k}$.

Now, since the columns of L and R have the properties described above and $T(V_a M_k V_a) \subset W_a M_m W_a$ ($1 \leq a \leq s$) and $W_a W_b = 0$ ($a \neq b$) then $(\text{tr}(T(l_i \bar{l}_i^t) r_j \bar{r}_j^t))_{k \times m}$ is a direct sum of the matrices $(\text{tr}(T(l_i \bar{l}_i^t) r_j \bar{r}_j^t))_{k_a \times m_a}$ with $k_1 + \dots + k_{a-1} + 1 \leq i \leq k_1 + \dots + k_a$ and $m_1 + \dots + m_{a-1} + 1 \leq j \leq m_1 + \dots + m_a$ (see the definition of the direct sum in the end of the introduction).

Notice that if $T : V_a M_k V_a \rightarrow W_a M_m W_a$ has support (total support), for every a , then the matrices $(\text{tr}(T(l_i \bar{l}_i^t) r_j \bar{r}_j^t))_{k_a \times m_a}$ with $k_1 + \dots + k_{a-1} + 1 \leq i \leq k_1 + \dots + k_a$ and $m_1 + \dots + m_{a-1} + 1 \leq j \leq m_1 + \dots + m_a$ have support (total support), by definition 2.2. Since $\frac{m_a}{k_a} = \frac{m}{k}$, for every a , then the matrix $(\text{tr}(T(l_i \bar{l}_i^t) r_j \bar{r}_j^t))_{k \times m}$ has support (total support), by item 8 of lemma 1.5. Therefore B has support (total support) by definition 1.2.

(1) Next, $\frac{\sqrt{k}}{\sqrt{m}} = \text{tr}((\frac{\sqrt{k}}{\sqrt{m}} Id) s_{j,n} \bar{s}_{j,n}^t) = \sum_{i=1}^k \text{tr}(Y_n T(X_n m_{i,n} \bar{m}_{i,n}^t X_n^*) Y_n^* s_{j,n} \bar{s}_{j,n}^t)$, since $Y_n T(X_n X_n^*) Y_n^* = \frac{\sqrt{k}}{\sqrt{m}} Id$, by item 1 of lemma 2.7.

Therefore, every entry of C_n is smaller or equal to $\frac{\sqrt{k}}{\sqrt{m}}$ and $\sigma'(C_n) \leq \left(\frac{\sqrt{k}}{\sqrt{m}}\right)^{mk}$, for every permutation $\sigma' \in S_{mk}$. Hence, by item 1 of lemma 1.7, $\sigma'(A_n) \sigma'(B_n) = \sigma'(C_n) \leq \left(\frac{\sqrt{k}}{\sqrt{m}}\right)^{mk}$, for every $\sigma' \in S_{mk}$.

Since $T : V_a M_k V_a \rightarrow W_a M_m W_a$ has support and $\frac{m_a}{k_a} = \frac{m}{k}$, for every a , then B has support. Hence, there is $\sigma \in S_{mk}$, such that $\sigma(B) > 0$, and there is $N > 0$, such that if $n > N$, $\sigma(B_n) > \frac{\sigma(B)}{2}$. Hence, for $n > N$, $\sigma(A_n) \leq 2\sigma(B)^{-1} \left(\frac{\sqrt{k}}{\sqrt{m}}\right)^{mk}$.

Now, by item 3 of lemma 1.7, $\sigma(A_n) = (\prod_{i=1}^k x_{i,n}^2)^m (\prod_{j=1}^m y_{j,n}^2)^k = \det(X_n X_n^*)^m \det(Y_n Y_n^*)^k = \det(X_n X_n^* \otimes Y_n Y_n^*)$. Since, by item 3 of lemma 2.7, $\det(X_n \otimes Y_n) > 0$ then $\det(X_n X_n^* \otimes Y_n Y_n^*) = \det(X_n \otimes Y_n)^2$.

Thus, $\det(X_n \otimes Y_n)^2 \leq 2\sigma(B)^{-1} \left(\frac{\sqrt{k}}{\sqrt{m}}\right)^{mk}$, for $n > N$. Recall that we simplified the notation in the begining of this lemma. Thus, we have just proved that there is a bounded subsequence of $(\det(X_n \otimes Y_n))_{n \in \mathbb{N}}$. Since the entire sequence is increasing, by item 3 of lemma 2.7, then the entire sequence is bounded. So, by item 4 of lemma 2.7, $T_1 : M_k \rightarrow M_m$ is doubly stochastic.

(2) We have just seen that $(\det(X_n \otimes Y_n))_{n \in \mathbb{N}}$ is bounded and increasing then $\lim_{n \rightarrow \infty} \det(X_n \otimes Y_n) = L > 0$ and $\lim_{n \rightarrow \infty} \sigma(A_n) = L^2$.

Let $\text{tr}(T(l_i \bar{l}_i^t) r_j \bar{r}_j^t)$ be any non-null entry of B . Since $T : V_a M_k V_a \rightarrow W_a M_m W_a$ has total support and $\frac{m_a}{k_a} = \frac{m}{k}$, for every a , then B has total support. Thus, there is a permutation $\sigma \in S_{mk}$ such that $\sigma(B) > 0$ and $\text{tr}(T(l_i \bar{l}_i^t) r_j \bar{r}_j^t)$ is one of the factors of $\sigma(B)$, by definition 1.1.

Notice that $\text{tr}(T_1(m_i \bar{m}_i^t) s_j \bar{s}_j^t)$ is a factor of $\sigma(C)$, since it occupies the same position of that $\text{tr}(T(l_i \bar{l}_i^t) r_j \bar{r}_j^t)$ in B .

Since $0 \neq L^2 \sigma(B) = \lim_{n \rightarrow \infty} \sigma(A_n) \sigma(B_n) = \sigma(C)$ and $\text{tr}(T_1(m_i \bar{m}_i^t) s_j \bar{s}_j^t)$ is a factor of $\sigma(C)$ then $\text{tr}(T_1(m_i \bar{m}_i^t) s_j \bar{s}_j^t) \neq 0$. Therefore,

$$\lim_{n \rightarrow \infty} x_{i,n} y_{j,n} = \lim_{n \rightarrow \infty} \text{tr}(Y_n T(X_n m_{i,n} \bar{m}_{i,n}^t X_n^*) Y_n^* s_{j,n} \bar{s}_{j,n}^t)^{\frac{1}{2}} \text{tr}(T(l_{i,n} \bar{l}_{i,n}^t) r_{j,n} \bar{r}_{j,n}^t)^{-\frac{1}{2}} = \text{tr}(T_1(m_i \bar{m}_i^t) s_j \bar{s}_j^t)^{\frac{1}{2}} \text{tr}(T(l_i \bar{l}_i^t) r_j \bar{r}_j^t)^{-\frac{1}{2}} \neq 0.$$

So $(x_{i,n} y_{j,n})_{k \times m}$ is a rank 1 matrix whose entries are positive and converge to positive limits, whenever the corresponding entries of the matrix $(\text{tr}(T(l_i \bar{l}_i^t) r_j \bar{r}_j^t))_{k \times m}$ are not zero. Since $(\text{tr}(T(l_i \bar{l}_i^t) r_j \bar{r}_j^t))_{k \times m}$ has total support then there are sequences of positive numbers $(x'_{i,n})_{n \in \mathbb{N}}$ $(y'_{j,n})_{n \in \mathbb{N}}$ converging to positive limits $x'_i > 0$, $y'_j > 0$ such that $x'_{i,n} y'_{j,n} = x_{i,n} y_{j,n}$ for every i, j, n , by lemma 1.6.

Define $X'_n = L_n(\text{diag}(x'_{1,n}, \dots, x'_{k,n})) M_n^*$, $Y'_n = S_n(\text{diag}(y'_{1,n}, \dots, y'_{m,n})) R_n^*$. Notice that $\lim_{n \rightarrow \infty} X'_n = X'$ and $\lim_{n \rightarrow \infty} Y'_n = Y'$, where $X' = L(\text{diag}(x'_1, \dots, x'_k)) M^*$ and $Y' = S(\text{diag}(y'_1, \dots, y'_m)) R^*$ are invertible matrices.

Finally, notice that $\text{tr}(Y_n T(X_n m_{i,n} \bar{m}_{p,n}^t X_n^*) Y_n^* s_{j,n} \bar{s}_{q,n}^t) = x_{i,n} x_{p,n} y_{j,n} y_{q,n} \text{tr}(T(l_{i,n} \bar{l}_{p,n}^t) r_{j,n} \bar{r}_{q,n}^t) = x'_{i,n} y'_{j,n} x'_{p,n} y'_{q,n} \text{tr}(T(l_{i,n} \bar{l}_{p,n}^t) r_{j,n} \bar{r}_{q,n}^t) = \text{tr}(Y'_n T(X'_n m_{i,n} \bar{m}_{p,n}^t X_n'^*) Y_n'^* s_{j,n} \bar{s}_{q,n}^t)$ for $\{i, p\} \subset \{1, \dots, k\}$ and $\{j, q\} \subset \{1, \dots, m\}$. Therefore, $Y_n T(X_n X X_n^*) Y_n^* = Y'_n T(X'_n X X_n'^*) Y_n'^*$, for every $X \in M_k$, and

$$T_1(X) = \lim_{n \rightarrow \infty} Y_n T(X_n X X_n^*) Y_n^* = \lim_{n \rightarrow \infty} Y'_n T(X'_n X X_n'^*) Y_n'^* = Y' T(X' X X'^*) Y'^*.$$

□

Corollary 2.10. *Let $T : M_k \rightarrow M_m$ be a positive map such that $T(\text{Id})$, $T^*(\text{Id})$ are positive definite Hermitian matrices. Let $X_n, A_n \in M_k, Y_n \in M_m$ be the matrices defined in algorithm 2.6. Then the following conditions are equivalent:*

- (1) T has support,
- (2) $(\det(X_n \otimes Y_n))_{n \in \mathbb{N}}$ is a bounded sequence,
- (3) $\lim_{n \rightarrow \infty} A_n = \frac{\text{Id}}{\sqrt{k}}$.

Proof. In the proof of item 1 of lemma 2.9, we saw that (1) implies (2) (Choose $V_1 = \text{Id}$ and $W_1 = \text{Id}$). In the proof of item 4 of lemma 2.7, we saw that (2) implies (3).

Now, let us prove that (3) implies (1). We also saw in the proof of item 4 of lemma 2.7 that if $\lim_{n \rightarrow \infty} A_n = \frac{\text{Id}}{\sqrt{k}}$ then the limit points of the sequence of positive maps $(Y_n T(X_n X X_n^*) Y_n^*)_{n \in \mathbb{N}}$ are doubly stochastic.

Thus, there is a sequence of positive maps $T_i : M_k \rightarrow M_m$ equivalent to T (i.e. $T_i(X) = Y_{n_i} T(X_{n_i} X X_{n_i}^*) Y_{n_i}^*$) converging to a doubly stochastic map $S : M_k \rightarrow M_m$, which has support (See 2.5). Notice that if any T_i has support then T should also have support, by lemma 2.3.

Let us assume by contradiction that every T_i does not have support. Recall that if $A \in P_k$ and $U \in P_k$ is the orthogonal projection onto $\Im(A)$ then $\Im(T_i(A)) = \Im(T_i(U))$, since $T_i : M_k \rightarrow M_m$ is a positive map. So, for each T_i , there is an orthogonal projection $U_i \in M_k$ such that $\text{rank}(U_i)m > \text{rank}(T_i(U_i))k$, by lemma 2.3.

Next, there is a subsequence $(U_j)_j$ of $(U_i)_i$ such that $\lim_j U_j = U$, $\text{rank}(U_j) = u$ and $\text{rank}(T_j(U_j)) = t$, for every j . Thus, $S(U) = \lim_j T_j(U_j)$ and $\text{rank}(U) = \text{tr}(U) = \lim_j \text{tr}(U_j) = u$.

Since the set of matrices with rank smaller or equal to t is closed then $\text{rank}(S(U)) \leq t$. Therefore, $\text{rank}(U)m = um > tk \geq \text{rank}(S(U))k$. This is a contradiction, since S has support. Thus, there is T_i with support and T has also support. \square

Theorem 2.11. *Let $T : M_k \rightarrow M_m$ be a positive map such that $T(\text{Id})$, $T^*(\text{Id})$ are positive definite Hermitian matrices. Then $T : M_k \rightarrow M_m$ is equivalent to a doubly stochastic map if and only if there are orthogonal projections $V_i \in M_k$, $W_i \in M_m$, $1 \leq i \leq s$, satisfying*

- (1) $\mathbb{C}^k = \bigoplus_{i=1}^s \Im(V_i)$, $\mathbb{C}^m = \bigoplus_{i=1}^s \Im(W_i)$,
- (2) $T(V_i M_k V_i) \subset W_i M_m W_i$,
- (3) $\text{rank}(X)\text{rank}(W_i) < \text{rank}(T(X))\text{rank}(V_i)$, if $X \in P_k \cap V_i M_k V_i$ and $0 < \text{rank}(X) < \text{rank}(V_i)$,
- (4) $\frac{\text{rank}(W_i)}{\text{rank}(V_i)} = \frac{m}{k}$, for every i .

Proof. First, let us assume that there are orthogonal projections $V_i \in M_k$, $W_i \in M_m$, $1 \leq i \leq s$, satisfying these four conditions. Let $X' \in M_k$ and $Y' \in M_m$ be invertible matrices such that $\Im(X'^{-1}V_i) \perp \Im(X'^{-1}V_j)$ and $\Im(Y'W_i) \perp \Im(Y'W_j)$, $i \neq j$. Let \tilde{V}_i, \tilde{W}_i be the orthogonal projections onto $\Im(X'^{-1}V_i)$, $\Im(Y'W_i)$, respectively.

Define $\tilde{T}(X) = Y'T(X'XX'^*)Y'^*$. Notice that $\tilde{T}(\tilde{V}_i M_k \tilde{V}_i) \subset \tilde{W}_i M_m \tilde{W}_i$. Moreover, if $X \in P_k \cap \tilde{V}_i M_k \tilde{V}_i$ and $0 < \text{rank}(X) < \text{rank}(\tilde{V}_i)$ then $\text{rank}(X)\text{rank}(\tilde{W}_i) < \text{rank}(\tilde{T}(X))\text{rank}(\tilde{V}_i)$. By remark 2.4, $\tilde{T} : \tilde{V}_i M_k \tilde{V}_i \rightarrow \tilde{W}_i M_m \tilde{W}_i$ has total support for every i . Now, since $\frac{\text{rank}(\tilde{W}_i)}{\text{rank}(\tilde{V}_i)} = \frac{m}{k}$, for every i , then \tilde{T} is equivalent to a doubly stochastic map, by lemma 2.9. Thus, T is equivalent to a doubly stochastic map.

For the converse, let $S : M_k \rightarrow M_m$ be a doubly stochastic map. We saw in 2.5 that S has total support. Thus, if there is $X \in P_k$ such that $\text{rank}(X)m = \text{rank}(S(X))k$ and $0 < \text{rank}(X) < k$ then $\Im(S(X^\perp)) = \Im(S(X)^\perp)$, by lemma 2.3. Therefore, $\text{rank}(X^\perp)m = \text{rank}(S(X)^\perp)k = \text{rank}(S(X^\perp))k$.

Let V be the orthogonal projection onto $\Im(X)$ and W the orthogonal projection onto $\Im(S(X))$. Since S is a positive map, $\Im(S(V)) = \Im(W)$ and $\Im(S(V^\perp)) = \Im(S(V)^\perp) = \Im(W^\perp)$ then $S(VM_k V) \subset WM_m W$ and $S(V^\perp M_k V^\perp) \subset W^\perp M_m W^\perp$. Hence, $S^*(WM_m W) \subset VM_k V$ and $S^*(W^\perp M_m W^\perp) \subset V^\perp M_k V^\perp$.

By definition of doubly stochastic map, $S(\frac{1}{\sqrt{k}}(V+V^\perp)) = \frac{1}{\sqrt{m}}(W+W^\perp)$ and $S^*(\frac{1}{\sqrt{m}}(W+W^\perp)) = \frac{1}{\sqrt{k}}(V+V^\perp)$, therefore $S(\frac{V}{\sqrt{k}}) = \frac{W}{\sqrt{m}}$, $S(\frac{V^\perp}{\sqrt{k}}) = \frac{W^\perp}{\sqrt{m}}$, $S^*(\frac{W}{\sqrt{m}}) = \frac{V}{\sqrt{k}}$ and $S^*(\frac{W^\perp}{\sqrt{m}}) = \frac{V^\perp}{\sqrt{k}}$.

Since $\sqrt{\text{rank}(V)}\sqrt{m} = \sqrt{\text{rank}(W)}\sqrt{k}$ and $\sqrt{\text{rank}(V^\perp)}\sqrt{m} = \sqrt{\text{rank}(W^\perp)}\sqrt{k}$ then $S(\frac{V}{\sqrt{\text{rank}(V)}}) = \frac{W}{\sqrt{\text{rank}(W)}}$, $S(\frac{V^\perp}{\sqrt{\text{rank}(V^\perp)}}) = \frac{W^\perp}{\sqrt{\text{rank}(W^\perp)}}$, $S^*(\frac{W}{\sqrt{\text{rank}(W)}}) = \frac{V}{\sqrt{\text{rank}(V)}}$ and $S^*(\frac{W^\perp}{\sqrt{\text{rank}(W^\perp)}}) = \frac{V^\perp}{\sqrt{\text{rank}(V^\perp)}}$.

Therefore, $S : VM_k V \rightarrow WM_m W$ and $S : V^\perp M_k V^\perp \rightarrow W^\perp M_m W^\perp$ are doubly stochastic maps. Now, we can use induction on the rank of V and V^\perp in order to find the subalgebras satisfying the conditions of this theorem.

Finally, if T is a positive map equivalent to S then we can easily find the required subalgebras satisfying the required conditions. \square

Theorem 2.12. *A positive map $T : M_k \rightarrow M_m$ is equivalent to a doubly stochastic map if and only if there are invertible matrices $X' \in M_k$, $Y' \in M_m$ such that $Y'T(X'XX'^*)Y'^*$ has total support and $T(Id)$, $T^*(Id)$ are positive definite Hermitian matrices.*

Proof. Since every doubly stochastic map has total support then the existence of $X' \in M_k$, $Y' \in M_m$ such that $Y'T(X'XX'^*)Y'^*$ has total support is necessary.

Now, since $T(Id)$, $T^*(Id)$ are positive definite Hermitian matrices and X' , Y' are invertible matrices then $Y'T(X'XX'^*)Y'^*$, $X'^*T^*(Y'^*Y')X'$ are positive definite Hermitian matrices. Next, if $Y'T(X'XX'^*)Y'^*$ has total support then there are invertible matrices $X'' \in M_k$ and $Y'' \in M_m$ such that $Y''Y'T(X'X''XX''^*X'^*)Y'^*Y''^*$ is doubly stochastic, by lemma 2.9 (choose $V_1 = Id$ and $W_1 = Id$). \square

Remark 2.13. *Since every doubly stochastic map has support by 2.5 then every positive map equivalent to a doubly stochastic map has also support by lemma 2.3. Thus, the condition of support is necessary for the equivalence of a positive map with a doubly stochastic one. However, the condition of total support is not necessary. For example, let $T : M_2 \rightarrow M_2$ be $T(X) = RXR$, where $R = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. This map is clearly equivalent to $Id : M_2 \rightarrow M_2$, however it does not have total support. Notice that if $\{e_1, e_2\}$ is the canonical basis of \mathbb{C}^2 then the matrix $(tr(T(e_i e_i^t) e_j e_j^t))_{2 \times 2}$ is equal to R , which does not have total support.*

3. APPLICATION TO QUANTUM INFORMATION THEORY: FILTER NORMAL FORM

Let $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m \simeq M_{km}$ be a positive semidefinite Hermitian matrix. Define the maps $F_A(X) = \sum_{i=1}^n A_i tr(B_i X)$ and $G_A(X) = \sum_{i=1}^n B_i tr(A_i X)$. We may assume without loss of generality that A_i, B_i are Hermitian for every i , since A is Hermitian. Notice that if A_i, B_i are Hermitian for every i then $G_A^* = F_A$ with respect to the trace inner product. Since A is positive semidefinite then F_A and G_A are positive maps (Actually, $G_A(X^t)$ is completely positive by Choi theorem [4], since $A = \sum_{i,j=1}^k e_i e_j^t \otimes G_A((e_i e_j^t)^t)$, where $\{e_1, \dots, e_k\}$ is the canonical basis of \mathbb{C}^k).

Theorem 3.1. *Let $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m \simeq M_{km}$ be a positive semidefinite Hermitian matrix such that $G_A(Id) \in M_m$ and $F_A(Id) \in M_k$ are positive definite Hermitian matrices. There are invertible matrices $X' \in M_k$, $Y' \in M_m$ such that $(X' \otimes Y')A(X' \otimes Y')^* = \sum_{i=1}^n C_i \otimes D_i$, $C_1 = \frac{Id}{\sqrt{k}}$, $D_1 = \frac{Id}{\sqrt{m}}$ and $tr(C_i C_j) = tr(D_i D_j) = 0$, for every $i \neq j$, if and only if there are invertible matrices $X'' \in M_k$, $Y'' \in M_m$ such that $Y''G_A(X''XX''^*)Y''^*$ has total support.*

Proof. The existence of these matrices X', Y' is equivalent to $G_{(X' \otimes Y')A(X' \otimes Y')^*}(\frac{Id}{\sqrt{k}}) = \frac{Id}{\sqrt{m}}$ and $F_{(X' \otimes Y')A(X' \otimes Y')^*}(\frac{Id}{\sqrt{m}}) = \frac{Id}{\sqrt{k}}$. Since $(X' \otimes Y')A(X' \otimes Y')^*$ is Hermitian then $G_{(X' \otimes Y')A(X' \otimes Y')^*}^* = F_{(X' \otimes Y')A(X' \otimes Y')^*}$.

Therefore the existence of X', Y' is equivalent to $G_{(X' \otimes Y')A(X' \otimes Y')^*}(X) = Y'G_A(X'^*XX')Y'^*$ being a doubly stochastic map, which is equivalent to the existence of invertible matrices $X'' \in M_k$, $Y'' \in M_m$ such that $Y''G_A(X''XX''^*)Y''^*$ has total support, by theorem 2.12. \square

Corollary 3.2. *Let $V_1 \in M_k$, $W_1 \in M_m$ be orthogonal projections. Define $V_2 = Id - V_1$, $W_2 = Id - W_1$ and $A = V_1 \otimes W_1 + V_2 \otimes W_2 \in M_k \otimes M_m$. Then A can be put in the filter normal form if and only if $rank(V_1)m = rank(W_1)k$.*

Proof. If A can be put in the filter normal form then $G_A : M_k \rightarrow M_m$ has support, by theorem 3.1 and remark 2.13. Notice that $G_A(V_i) = tr(V_i)W_i$, $i = 1, 2$. By lemma 2.3, $rank(V_i)m \leq rank(W_i)k$, $i = 1, 2$ and since $\sum_{i=1}^2 rank(V_i)m = mk = \sum_{i=1}^2 rank(W_i)k$ then $rank(V_i)m = rank(W_i)k$, $i = 1, 2$.

Now, for the converse, if $rank(V_1)m = rank(W_1)k$ then $rank(V_2)m = rank(W_2)k$. Since $G_A(V_i) = tr(V_i)W_i$, $i = 1, 2$, then $G_A(V_i M_k V_i) \subset W_i M_m W_i$, $i = 1, 2$. By remark 2.4, $G_A : V_i M_k V_i \rightarrow$

$W_i M_m W_i$, $i = 1, 2$, has total support. Thus, $G_A : M_k \rightarrow M_m$ is equivalent to a doubly stochastic map by lemma 2.9. Therefore, by theorem 3.1, A can be put in the filter normal form. \square

Theorem 3.3. *Let $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m \simeq M_{km}$ be a positive semidefinite Hermitian matrix. If $k \neq m$ and $\dim(\ker(A)) < \min\{k, m\}$ or if $k = m$ and $\dim(\ker(A)) < k - 1$ then there are invertible matrices $X' \in M_k$, $Y' \in M_m$ such that $(X' \otimes Y')A(X' \otimes Y')^* = \sum_{i=1}^n C_i \otimes D_i$, $C_1 = \frac{Id}{\sqrt{k}}$, $D_1 = \frac{Id}{\sqrt{m}}$ and $\text{tr}(C_i C_j) = \text{tr}(D_i D_j) = 0$, for every $i \neq j$.*

Proof. First, let $w \in \mathbb{C}^k$, $v \in \mathbb{C}^m$ have norm 1. Notice that $\text{tr}(G_A(Id)v\bar{v}^t) = \text{tr}(A(Id \otimes v\bar{v}^t)) > 0$ and $\text{tr}(w\bar{w}^t F_A(Id)) = \text{tr}(A(w\bar{w}^t \otimes Id)) > 0$, since $\text{tr}(A(Id \otimes v\bar{v}^t))$ is bigger or equal to the sum of the k smallest eigenvalues of A and $\text{tr}(A(w\bar{w}^t \otimes Id))$ is bigger or equal to the sum of the m smallest eigenvalues of A . Therefore $G_A(Id)$ and $F_A(Id)$ are positive definite Hermitian matrices.

Next, let $\{v_1, \dots, v_k\} \subset \mathbb{C}^k$ and $\{w_1, \dots, w_m\} \subset \mathbb{C}^m$ be any orthonormal bases. Consider the matrix $(\text{tr}(G_A(v_i \bar{v}_i^t) w_j \bar{w}_j^t)) \in M_{k \times m}$.

Since $\text{tr}(A(v_i \bar{v}_i^t \otimes w_j \bar{w}_j^t)) = \text{tr}(G_A(v_i \bar{v}_i^t) w_j \bar{w}_j^t)$ then the cardinality of $\{(i, j) \mid \text{tr}(G_A(v_i \bar{v}_i^t) w_j \bar{w}_j^t) = 0\}$ is smaller than $\min\{k, m\}$, if $k \neq m$, or smaller than $k - 1$, if $k = m$. By lemma 1.5, the matrix $(\text{tr}(G_A(v_i \bar{v}_i^t) w_j \bar{w}_j^t)) \in M_{k \times m}$ has total support. Therefore $G_A : M_k \rightarrow M_m$ has total support. By theorem 3.1, the result follows. \square

Theorem 3.4. *Let $A = \sum_{i=1}^n A_i \otimes B_i \in M_k \otimes M_m \simeq M_{km}$ be a positive semidefinite Hermitian matrix such that $G_A(Id)$ and $F_A(Id)$ are positive definite Hermitian matrices. If $\dim(\ker(A)) < \frac{\max\{k, m\}}{\min\{k, m\}}$ then there are invertible matrices $X' \in M_k$, $Y' \in M_m$ such that $(X' \otimes Y')A(X' \otimes Y')^* = \sum_{i=1}^n C_i \otimes D_i$, $C_1 = \frac{Id}{\sqrt{k}}$, $D_1 = \frac{Id}{\sqrt{m}}$ and $\text{tr}(C_i C_j) = \text{tr}(D_i D_j) = 0$, for every $i \neq j$.*

Proof. Let $\{v_1, \dots, v_k\} \subset \mathbb{C}^k$ and $\{w_1, \dots, w_m\} \subset \mathbb{C}^m$ be any orthonormal bases. Notice that $\text{tr}(G_A(v_i \bar{v}_i^t) w_j \bar{w}_j^t) = \text{tr}(A(v_i \bar{v}_i^t \otimes w_j \bar{w}_j^t)) = \text{tr}(v_i \bar{v}_i^t F_A(w_j \bar{w}_j^t))$.

Since $G_A(Id)$ and $F_A(Id)$ are positive definite then the matrix $(\text{tr}(G_A(v_i \bar{v}_i^t) w_j \bar{w}_j^t))_{k \times m}$ has no row or column identically zero.

Next, since $\dim(\ker(A)) < \frac{\max\{k, m\}}{\min\{k, m\}}$ then the cardinality of $\{(i, j) \mid \text{tr}(G_A(v_i \bar{v}_i^t) w_j \bar{w}_j^t) = 0\} < \frac{\max\{k, m\}}{\min\{k, m\}}$. Thus, by item 7 of lemma 1.5, the matrix $(\text{tr}(G_A(v_i \bar{v}_i^t) w_j \bar{w}_j^t))_{k \times m}$ has total support. Therefore, $G_A : M_k \rightarrow M_m$ has total support. By lemma 3.1, the result follows. \square

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